Problem A.6

Evaluate the integral (to 5 significant digits)

$$\int_{-1}^{5} \theta(2x-4)e^{-3x}\,dx$$

Solution

$\theta(x)$ represents the Heaviside function, which is defined by

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Start by making the following substitution.

$$u = 2x - 4 \qquad \rightarrow \qquad x = \frac{u+4}{2}$$

$$du = 2\,dx \qquad \rightarrow \qquad \frac{du}{2} = dx$$

Therefore,

$$\begin{aligned}
\int_{-1}^{5} \theta(2x-4)e^{-3x}\,dx &= \int_{2(-1)-4}^{2(5)-4} \theta(u)e^{-3(u+4)/2}\left(\frac{du}{2}\right) \\
&= \frac{1}{2}\int_{-6}^{6} \theta(u)e^{-3u/2-6}\,du \\
&= \frac{e^{-6}}{2}\int_{-6}^{6} \theta(u)e^{-3u/2}\,du \\
&= \frac{e^{-6}}{2}\left[\int_{-6}^{0} \theta(u)e^{-3u/2}\,du + \int_{0}^{6} \theta(u)e^{-3u/2}\,du\right] \\
&= \frac{e^{-6}}{2}\left[\int_{-6}^{0} (0)e^{-3u/2}\,du + \int_{0}^{6} (1)e^{-3u/2}\,du\right] \\
&= \frac{e^{-6}}{2}\int_{0}^{6} e^{-3u/2}\,du \\
&= \frac{e^{-6}}{2}\left(-\frac{2}{3}e^{-3u/2}\Big|_{0}^{6}\right) \\
&= -\frac{e^{-6}}{3}(e^{-9}-1) \\
&= \frac{e^{9}-1}{3e^{15}} \\
&\approx 0.00082615.
\end{aligned}$$